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Entry-length flow in a vertical cooled pipe

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A numerical solution is presented for predicting the flow and heat transfer due to free convection in the entry region of a cooled vertical pipe, open at both ends. Following Lighthill (1953) the Kármán–Pohlhausen method is used in this analysis. Velocity and temperature profiles are assumed satisfying the physical boundary conditions, and the integrated forms of the equations of motion and the equations themselves at the axis and the walls of the pipe are used to calculate the various parameters involved in the profiles assumed. Two cases of constant wall temperature and linearly decreasing temperature with height were considered. Cooling decelerates the fluid in the boundary layer that forms at the wall and the fluid in the core is accelerated by virtue of continuity. As cooling progresses, at Rayleigh number $O(10^3)$ the fluid in the boundary layer starts reversing at first and then assumes progressively increasing velocities. Graphs are presented for the development of the predicted velocity and temperature profiles and the other parameters involved, for a situation when the boundary layer fills the whole pipe.

1. Introduction

In his classic review on internal flows with body forces Ostrach (1964, chapter 4) writes, "Most of the early work on internal flows with body forces was of an experimental or semi-empirical nature." An extensive study of the latter type was made by Elenbaas (1942) for the natural convection heat transfer between two parallel plates heated to the same temperature. None of this work however, gave detailed information on the velocity and temperature distributions; in some cases the semi-empirical formulas derived from the experimental data! Schmidt (1951) demonstrated the successful application of natural convection flows in hollow passages in turbine blades for cooling. Ostrach (1954) calculated buoyancy forces on the fluid within the pipe relative to the cool fluid at the same level outside the pipe. Even in his case it seems better to regard the fluid in the pipe as moving under an over-all pressure gradient and the relative buoyancy forces within. Ostroumov (1958) presented an extensive treatment of natural convection in cylindrical channels. His results are in general the same as given by other workers on channel flows, presented rather more elegantly in terms of the Bessel and Neumann functions. To investigate the effect of confining walls on convection Lighthill (1953) analysed the flow in a closed-end cylindrical tube with walls at constant temperature and the body force acting towards the closed end. An orifice is assumed at the open end which supplies the fluid. The type of flow in the tube depends on the aspect ratio (l/R) for given Prandtl and Rayleigh numbers. For small values of the ratio (l/R) the flow is just like free convection about a flat plate, i.e. the effect of the confining walls is negligible if the boundary layer thickness is small compared with the radius of the tube. For larger aspect ratios this is no longer true. Lighthill used an integral method to obtain his results. For the similarity régime he found that the flow fills the entire tube only for one value of the aspect ratio (l/R). Several authors have adapted Lighthill's original idea and method of treatment of free convection in confined flows to slightly varied physical situations. Ostrach & Thornton (1958) studied the stagnation of fluids in closed-end tubes with a configuration similar to that of Lighthill except that the temperature was allowed to vary along the wall. Leslie (1959) found approximate solutions for the fluid flow and heat transfer in a heated cylinder, closed at the bottom and opening at the top into a reservoir of cool fluid which has been tilted at a small angle to the vertical. Tilting causes a small increase in heat transfer which is proportional to the square of $(l/a)\tan\phi$, where l/a is the length/radius ratio and ϕ is the angle of tilt. Hammitt (1958) made an analysis of natural convection in the open thermosyphon with internal heat generation. Bayley & Lock (1965) described a series of closely controlled experiments made to study the performance of closed thermosyphons. A theoretical analysis of the laminar boundary layer régime is given and comparisons made with experimental results. Martin (1967) has made predictions, supported by experiments, of the heat transfer due to natural convection in long externally cooled vertical cylinders with uniform wall temperature containing a heat-generating fluid in the laminar flow, having Prandtl numbers of unity or above. Takhar (1967) considered the entry length flow in a vertical heated (open) pipe. He found that at values of the Rayleigh number $O(10^3)$ the flow in the middle of the pipe nearly approached stagnation. His method of analysis failed to produce satisfactory results before the boundary layer filled the entire pipe.

In the present problem an attempt is made to study the flow in the entrance region of a vertical pipe (open at both ends), which is cooled with (*a*) constant temperature at the wall, (*b*) wall temperature decreasing linearly as a function of the vertical height. It is assumed that coefficients of kinematic viscosity and thermal conductivity are effectively constant and the Boussinesq approximation holds. Following Lighthill the Kármán–Pohlhausen method is used in this analysis. Velocity and temperature profiles are assumed, satisfying the physical boundary conditions and the integrated forms of the equations of motion, and the equations themselves at the axis and the walls of the pipe are used to calculate the various parameters involved in the profiles assumed. It is found that in this problem the boundary layer fills the entire pipe giving way to a fully developed flow. A reversed flow is predicted for values of the cooling Rayleigh number $O(10^3)$. Graphs are presented for temperature and velocity profiles and other parameters used in these profiles for a situation when the boundary layer fills the whole pipe.

In the case of the heated pipe the flow is expected to reverse in the core region, whereas in the case of the cooled pipe the flow is predicted to reverse in the boundary layer. It may be because the Kármán–Pohlhausen method seems to be sensitive to the core region, that the present method of analysis fails to give satisfactory results in the case of a heated pipe. It is hoped that the present paper may prove useful in a better understanding of the engineering problems on free convection in the entrance region of pipes.

In the present paper no over-simplification of the governing equations has been achieved by setting Prandtl number equal to infinity (as Lighthill did except for the extreme cases of his problem). Instead the equations have been solved for Prandtl number equal to unity. The velocity profile assumed is scaled through the parameter A, while the parameter D allows for the vertical displacement of velocity outside the boundary layer; the Reynolds number at entry is also set equal to unity. A parabolic profile is assumed for the temperature distribution in the boundary layer. Also the momentum and thermal boundary layer thicknesses are assumed to be equal.

2. Theory

The equations of motion of this problem are similar to the ordinary differential equations of free convection except that the pressure no longer takes the hydrostatic value. The flow is assumed to be the boundary layer type flow, therefore one is justified in neglecting the gradient of a quantity along the pipe compared with its gradient along the radius. With these approximations the equations of conservation of mass, momentum and heat, in cylindrical polar co-ordinates, with x increasing along the pipe, reduce to

$$\frac{\partial u}{\partial x}+\frac{1}{r}\frac{\partial (rv)}{\partial r} = 0, \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial r} = -\left(\frac{1}{\rho}\frac{\partial p}{\partial x}+g\right)+\nu\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right)-\beta g(T_w-T), \tag{2}$$

$$O = -\frac{1}{\rho}\frac{\partial p}{\partial r}, \tag{3}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial r} = \kappa\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}\right), \tag{4}$$

where u, v are the axial and radial velocities, $\nu = \mu/\rho$ and μ the kinematic and dynamic viscosities, κ the thermal diffusivity and β the coefficient of cubic expansion.

The wall temperature T_w may be written as

$$T_w = T_0-\Delta T\,\Theta(X). \tag{5}$$

where T_0 is the temperature of the fluid at entry.

Variations of all the physical properties are ignored. Density changes are considered only in so far as they give rise to the buoyancy term. Viscous dissipation and the work done against the gravity field are ignored. The scale temperature $\Delta T = (T_w-T_0)$ defines a cooling Rayleigh number

$$Ra = \frac{\beta g a^3 \Delta T}{\kappa\nu}. \tag{6}$$

We make the transformations:

$$\left.\begin{aligned} r &= aR, \quad x = aX, \quad u = (\kappa/a)U, \quad v = (\kappa/a)V, \\ \Gamma\frac{\kappa\nu}{a^3} &= -\left(\frac{1}{\rho}\frac{\partial p}{\partial x}+g\right), \quad T = T_w+\frac{\Delta T}{Ra}\theta, \end{aligned}\right\} \tag{7}$$

where the capitals denote the non-dimensional quantities and a is the radius of the pipe. The equations (1)–(4) reduce to

$$\frac{\partial U}{\partial X}+\frac{1}{R}\frac{\partial(RV)}{\partial R} = 0, \tag{8}$$

$$\frac{1}{\sigma}\left(U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial R}\right) = \Gamma+\left(\frac{\partial^2 U}{\partial R^2}+\frac{1}{R}\frac{\partial U}{\partial R}\right)+\theta, \tag{9}$$

$$O = \partial P/\partial R, \tag{10}$$

where σ is the Prandtl number.

Using (5) the heat equation, (4) reduces to

$$-RaU\frac{d[\Theta(X)]}{dX}+\left(U\frac{\partial\theta}{\partial X}+V\frac{\partial\theta}{\partial R}\right) = \left(\frac{\partial^2\theta}{\partial R^2}+\frac{1}{R}\frac{\partial\theta}{\partial R}\right). \tag{11}$$

Two cases arise depending upon the way the pipe wall is cooled.

Case 1. When the wall temperature is kept constant, (11) reduces to

$$U\frac{\partial\theta}{\partial X}+V\frac{\partial\theta}{\partial R} = \frac{\partial^2\theta}{\partial R^2}+\frac{1}{R}\frac{\partial\theta}{\partial R}. \tag{12}$$

Case 2. When the wall temperature decreases linearly as a function of X, i.e.

$$T_w = T_0-\Delta T X, \tag{13}$$

equation (11) now reduces to

$$-RaU+\left(U\frac{\partial\theta}{\partial X}+V\frac{\partial\theta}{\partial R}\right) = \frac{\partial^2\theta}{\partial R^2}+\frac{1}{R}\frac{\partial\theta}{\partial R}. \tag{14}$$

For a uniform flow into the pipe the boundary conditions are

$$\left.\begin{aligned} R &= 1, \quad U = 0, \quad V = 0, \quad \theta = 0, \quad \text{all} \quad X > 0; \\ X &= 0, \quad U = U_0, \quad V = 0, \quad \theta = Ra, \quad \text{all} \quad R < 1, \end{aligned}\right\} \tag{15}$$

where U_0 is the non-dimensional axial velocity at entry, which is also equal to the Reynolds number at entry since the Prandtl number is equal to unity ($U_0/\sigma = Re$).

Equations (8), (9) and (11), when integrated over a cross-section subject to the boundary conditions (15), reduce to

$$\int_0^1 RU\,dR = \tfrac{1}{2}U_0, \tag{16}$$

$$\frac{1}{\sigma}\frac{d}{dX}\int_0^1 RU^2\,dR = \tfrac{1}{2}\Gamma+\left(\frac{\partial U}{\partial R}\right)_1+\int_0^1 R\theta\,dR, \tag{17}$$

$$-\tfrac{1}{2}RaU_0\frac{d[\Theta(X)]}{dX}+\frac{d}{dX}\int_0^1 RU\theta\,dR = \left(\frac{\partial\theta}{\partial R}\right)_1. \tag{18}$$

The equations (8), (9) and (12) themselves at the axis and the wall of the pipe reduce to

$$R = 0: \quad \left.\begin{aligned} \frac{1}{\sigma} U \frac{\partial U}{\partial X} &= \Gamma + \frac{\partial^2 U}{\partial R^2} + \frac{1}{R}\frac{\partial U}{\partial R} + \theta, \\ U \frac{\partial \theta}{\partial X} &= \frac{\partial^2 \theta}{\partial R^2} + \frac{1}{R}\frac{\partial \theta}{\partial R}; \end{aligned}\right\} \tag{19}$$

$$R = 1: \quad \left.\begin{aligned} O &= \Gamma + \frac{\partial^2 U}{\partial R^2} + \frac{1}{R}\frac{\partial U}{\partial R}, \\ O &= \frac{\partial^2 \theta}{\partial R^2} + \frac{1}{R}\frac{\partial \theta}{\partial R}. \end{aligned}\right\} \tag{20}$$

Assume Kármán–Pohlhausen type profiles for U and θ in terms of $Y = 1-R$ and $\Delta = 1-S$, satisfying the physical boundary conditions

$$U = \begin{bmatrix} AU_0 \quad (\Delta < Y < 1), \\ AU_0\left\{1-\left(1-\frac{Y}{\Delta}\right)^2\left(1-\frac{DY}{\Delta}\right)\right\} \quad (0 < Y < \Delta), \end{bmatrix} \tag{21}$$

$$\theta = \begin{bmatrix} Ra \quad (\Delta < Y < 1), \\ Ra\left\{1-\left(1-\frac{Y}{\Delta}\right)^2\right\} \quad (0 < Y < \Delta). \end{bmatrix} \tag{22}$$

This corresponds to a boundary layer of non-dimensional thickness Δ on the wall of the pipe enclosing a potential core of radius $S = 1-\Delta$. Putting these values of U and θ in (16)–(20) we get:

Case 1.
$$A[5(6-4\Delta+\Delta^2)+D\Delta(5-2\Delta)] = 30, \tag{23}$$

$$\frac{d}{d\Delta}[A\{14(15-14\Delta+4\Delta^2)+D\Delta(21-10\Delta)\}] = \left[-\frac{840}{U_0\Delta}\right]\frac{dX}{d\Delta}, \tag{24}$$

$$\frac{U_0^2}{840\sigma}\frac{d}{d\Delta}[A^2\{14(15-28\Delta+8\Delta^2)+4D\Delta(21-10\Delta)+D^2\Delta(8-3\Delta)\}] = \left[-\frac{Ra}{12}(4-\Delta)\Delta-\frac{U_0 A}{\Delta}(2+D)\right]\frac{dX}{d\Delta}. \tag{25}$$

Initial boundary conditions are $A = 1$, $D = 0$ at $X = 0$, $\Delta = 0$. Obviously $\Delta = 0$ is a singularity for the above equations. We shall try to find out the values of A and D in the vicinity of the entry edge by attempting a series solution in terms of the boundary layer thickness.

We suppose
$$A = [1+a_1\Delta+a_2\Delta^2+a_3\Delta^3+\dots]. \tag{26}$$

This leads (after simplification) to

$$A = [1+\tfrac{2}{3}\Delta-\tfrac{1}{30}\Delta^2+\dots], \tag{27}$$

$$D = 6[\tfrac{14}{45}\Delta+\dots], \tag{28}$$

$$X = \tfrac{1}{30}[\Delta^2+\tfrac{91}{90}\Delta^3+\dots]. \tag{29}$$

These values enable the integration of A, D and X in terms of Δ to be started in the vicinity of the entry edge and it was continued by the Runge–Kutta method on the Atlas Computer.

Case 2. When the wall temperature decreases linearly as a function of X, the treatment is exactly similar to the one outlined in case 1. We get, on simplification

$$A[5(6-4\Delta+\Delta^2)+D\Delta(5-2\Delta)] = 30, \tag{23}$$

$$\frac{d}{d\Delta}[A\{14(15-14\Delta+4\Delta^2)+D\Delta(21-10\Delta)\}] = \left[-\frac{840}{U_0\Delta}+210\right]\frac{dX}{d\Delta} \tag{30}$$

and (25) as given above.

Once again we suppose that (26) holds and proceed as before to get (after simplification), (27), (28) and

$$X = \tfrac{1}{30}[\Delta^2+\Delta^3\tfrac{106}{90}+\dots]. \tag{31}$$

We thus know A, D and X in terms of the boundary-layer thickness Δ. As before the three reduced differential equations are solved by the Runge–Kutta method on our Atlas Computer. It may be interesting to note that the characteristic heat-transfer properties can be calculated from the present analysis as follows:

$$\text{Nusselt number} = \frac{\begin{array}{c}\text{Rate of heat transfer per unit}\\ \text{area of the pipe wall} \times \text{pipe diameter}\end{array}}{\begin{array}{c}\text{Characteristic temperature difference in}\\ \text{the main direction of conduction}\end{array}} = \frac{4}{\Delta}; \tag{32}$$

$$\text{the heat flux at the wall} = (\partial\theta/\partial Y)_0 = 2Ra/\Delta; \tag{33}$$

the skin friction at the wall is given by

$$\left(\frac{\partial U}{\partial Y}\right)_0 = AU_0\frac{2+D}{\Delta}. \tag{34}$$

3. Discussion

The cooling of the wall decelerates the fluid in the boundary layer close to the wall and the fluid in the core is accelerated due to continuity. At values of the cooling Rayleigh number $O(10^3)$ the retardation of the fluid in the boundary layer is so great as to cause a reversal of flow in the region close to the wall. The core velocity would increase correspondingly. In free convection problems the adverse pressure gradient is confined to the boundary layer produced by the buoyancy forces, whereas in the forced convection problems it takes place in the mainstream as well. The results predicted on the basis of the present analysis are more or less similar for the two modes of cooling depicted through figures 1–5 and 6–10 respectively, for a situation when the boundary layer fills the entire pipe.

Figures 1 and 6 illustrate the velocity profiles in cases 1 and 2 at different Rayleigh numbers. The axial heights reached for attaining these velocities for a situation when the boundary layer fills the entire pipe are shown in figures 3 and 8, which themselves depict the development of the non-dimensional boundary layer thickness Δ with the non-dimensional axial distance X at different Rayleigh numbers. Fluid in the boundary layer is decelerated due to cooling while the fluid in the core is accelerated. With increasing Rayleigh

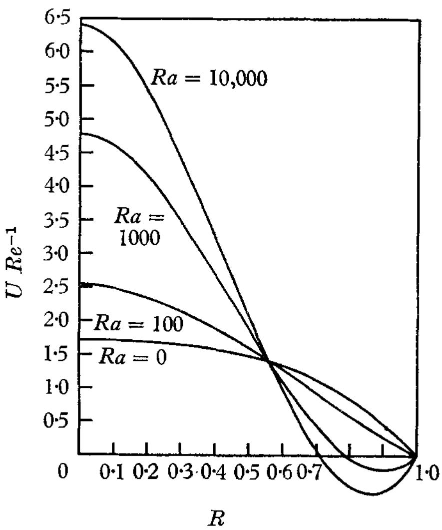

FIGURE 1

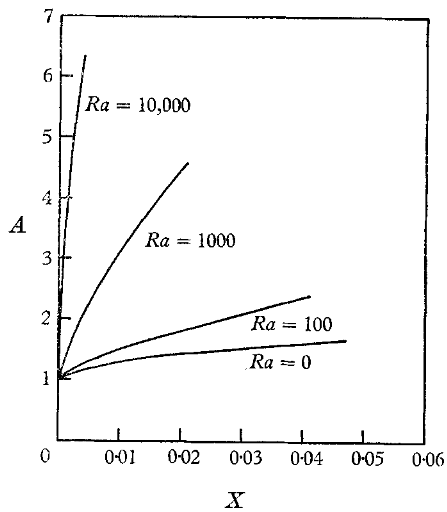

FIGURE 2

FIGURE 1. Velocity profiles at different Rayleigh numbers with constant wall temperature.

FIGURE 2. Development of the non-dimensional velocity parameter A with the non-dimensional axial distance X at different Rayleigh numbers with constant wall temperature.

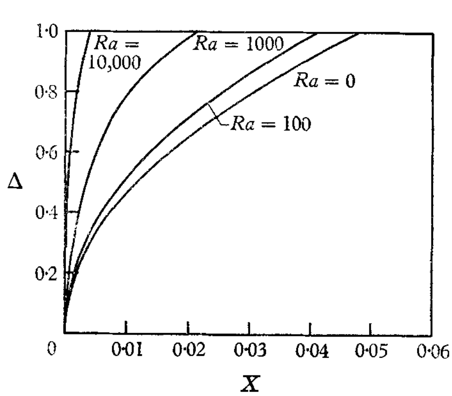

FIGURE 3

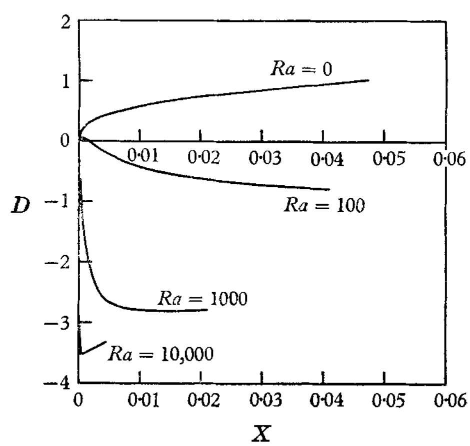

FIGURE 4

FIGURE 3. Development of the non-dimensional boundary-layer thickness Δ with the non-dimensional axial distance X at different Rayleigh numbers with constant wall temperature.

FIGURE 4. Development of shape parameter D of the velocity profile at different Rayleigh numbers with constant wall temperature.

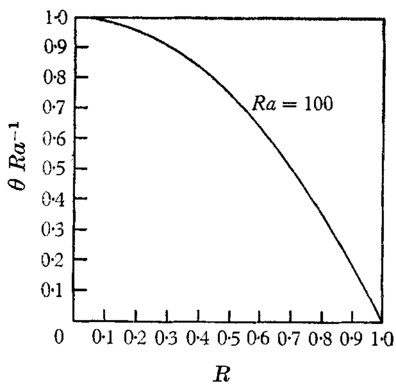

FIGURE 5

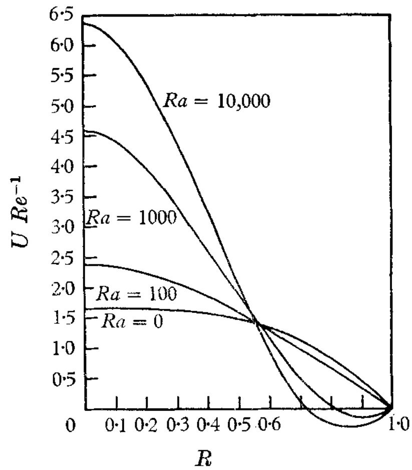

FIGURE 6

FIGURE 5. Temperature profile for $Ra = 100$ at $X = 0{\cdot}041$ with constant wall temperature.

FIGURE 6. Velocity profiles at different Rayleigh numbers with linearly decreasing wall temperature.

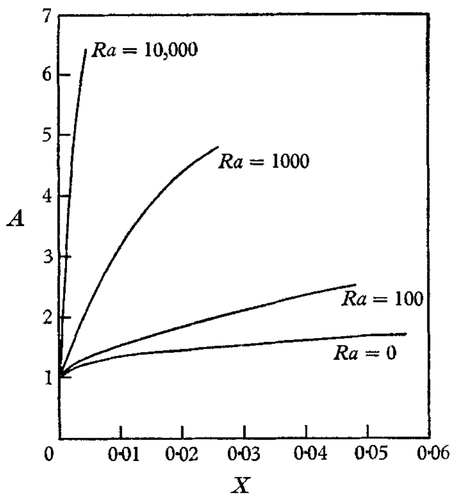

FIGURE 7

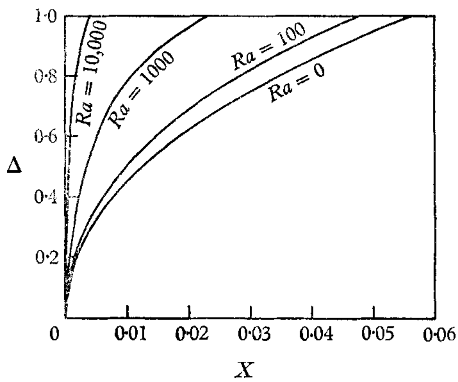

FIGURE 8

FIGURE 7. Development of the non-dimensional velocity parameter A with the non-dimensional axial distance X at different Rayleigh numbers with linearly decreasing wall temperature.

FIGURE 8. Development of the non-dimensional boundary-layer thickness Δ with the non-dimensional axial distance X at different Rayleigh numbers with linearly decreasing wall temperature.

numbers the boundary layer fills the whole pipe for gradually decreasing axial heights. Figures 2 and 7 illustrate the development of the non-dimensional velocity scaling factor $A = U Re^{-1}$ with the non-dimensional axial distance X for different Rayleigh numbers. Figures 4 and 9 show the development of the boundary layer thickness scaling factor D with the non-dimensional axial distance X for different Rayleigh numbers. It is noticed that this parameter starts assuming negative values for Rayleigh numbers $O(10^2)$. Figures 5 and 10 illustrate the development of the temperature profile for $Ra = 100$ only. These two

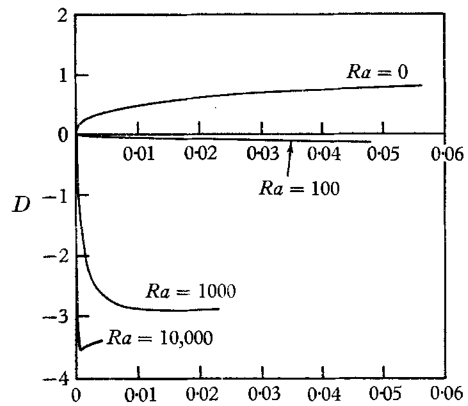

FIGURE 9

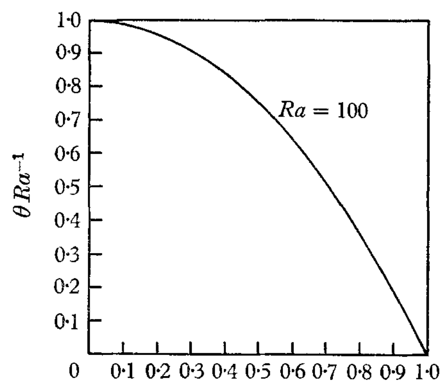

FIGURE 10

FIGURE 9. Development of the shape parameter D of the velocity profile at different Rayleigh numbers with linearly decreasing wall temperature.

FIGURE 10. Temperature profile for $Ra = 100$ at $X = 0{\cdot}048$ with linearly decreasing wall temperature.

figures are similar except that these profiles are reached at different axial heights in the two cases. Temperature profiles for other Rayleigh numbers would be similar in shape though not in scale for both the cases considered.

As far as I know, experimental verification of the predictions made in this paper is not available yet.

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REFERENCES

BAYLEY, F. J. & LOCK, G. S. H. 1965 *ASME J. Heat Transfer*, **87**, 30–39.

ELENBAAS, W. 1942 *Physica*, **9**, 1–28.

HAMMITT, F. G. 1958 *ASME Paper*, no. 58-SA-30 (Semi-Annual Meeting, Detroit, Michigan).

LESLIE, F. M. 1959 *J. Fluid Mech.* **7**, 115–27.

LIGHTHILL, M. J. 1953 *Q. J. Mech. Appl. Math.* **6**, 398–439.

MARTIN, B. W. 1967 *Proc. R. Soc.* A **301**, 327–41.

OSTRACH, S. 1954 *NACA Tech. Note* 3141.

OSTRACH, S. 1964 *High Speed Aerodynamics and Jet Propulsion.* Vol. IV. *Theory of Laminar Flows.* Ed. by F. K. Moore. Oxford University Press.

OSTRACH, S. & THORNTON, P. R. 1958 *Trans. ASME,* **80**, 363–6.

OSTROUMOV, G. A. 1958 *Translation NACA Tech. Mem.* 1407.

SCHMIDT, E. H. W. 1951 *Proceedings of General Discussion on Heat Transfer, Section IV,* 361–3. Institution of Mechanical Engineers.

TAKHAR, H. S. 1967 *Proceedings of Japanese Society of Mechanical Engineers, Semi-International Symposium Paper* no. 201, 1–8.